Chapter 14
Areas Under Curves

The material in this chapter is from Achievement Standard 90636 (Calculus 3.2) 'Integrate functions and use integrals to solve problems', and contains:

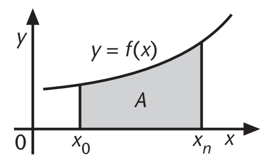
- Finding areas under curves of functions (with diagrams provided) of the type
  - ax^n, where n is a real number, including $n = -1$
  - polynomials
  - exponential functions of the form ae^{bx+c} (base e only)
  - trigonometric functions
  - rational functions such as $\dfrac{ax + b}{x}$.
- Areas between graphs of polynomial functions.
- Finding areas associated with sums, products and combinations of the standard functions.
- Using a variety of integration techniques to solve problems involving areas between graphs of functions other than polynomials.

Introduction

In this chapter the applications of **definite integrals** to areas will be considered, with some practical applications.

The study of integration first arose because mathematicians needed to calculate areas between curves and straight lines. Integration was originally developed quite independently of differentiation, and the discovery by G. Leibnitz (1646–1716) and I. Newton (1642–1727) that the two concepts are related was an extremely valuable one. In the following discussion, it will be shown that areas between curves and straight lines can be calculated using definite integrals.

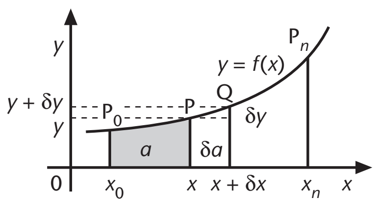
Areas Under Curves Using Definite Integrals



Let $y = f(x)$ be a function whose graph is a smooth and continuous curve lying above the x-axis.

Let A be the area of the shaded region between the curve and the x-axis, from $x = x_0$ to $x = x_n$.

The relationship between the area, A, and the definite integral, $\int_{x_0}^{x_n} f(x)\,dx$, will now be established.



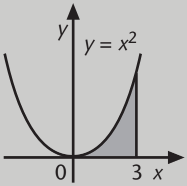
Let P (x, y) be any point on the curve. Suppose P was originally at P_0 (x_0, y_0) and finally takes up the position at P_n (x_n, y_n) as it moves along the curve. As the point moves from P_0 to a general position P, suppose it passes over an area a, which is drawn shaded.

If P moved a small distance further along the curve, to Q $(x + \delta x, y + \delta y)$ say, the small horizontal increment is denoted by δx, and the corresponding small vertical increment in its position is denoted by δy. The increment in area is denoted by δa, and this is the area between the curve and the x-axis from x to $x + \delta x$.

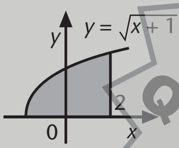
Activity 15.1: Volumes of revolution

1. If the shaded areas are rotated about the x-axis through 360°, find the volumes of the solids of revolution:

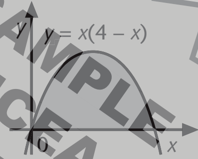
a.



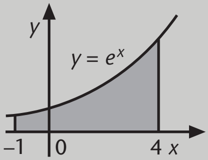
b.



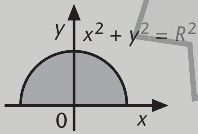
c.



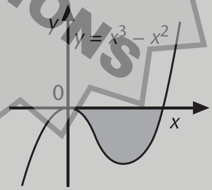
d.



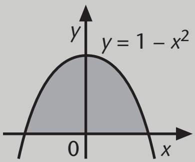
e.



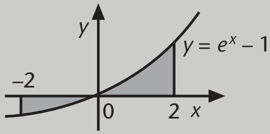
f.



g.

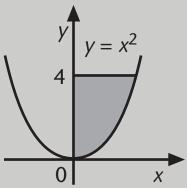


h.

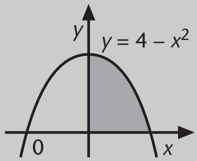


2. If the shaded areas are rotated about the y-axis through 360°, find the volumes of the solids of revolution.

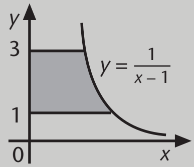
a.



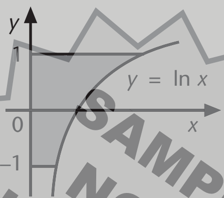
b.



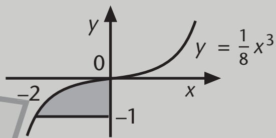
c.



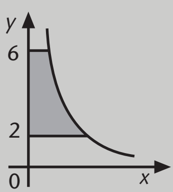
d.

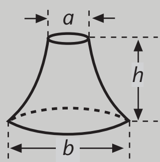


e.



3. Amy is designing a new brass candlestick. She has selected the rectangular hyperbola $xy = 12$ and rotated the shaded area about the y-axis through 360°, to obtain a volume of revolution. (The hole for the candle will be drilled out later.) Find the volume of the shape shown, and the dimensions a, h and b if the units are cm.





*Some answers are not graded as **A, M, E** because those questions are not directly assessed in the Achievement Standards, but are helpful in preparing the student for the Achievement Standards.*

Activity 1.1: Basic trigonometry (page 8)

1. **a.** $\frac{-1}{\sqrt{2}}$ **b.** $\frac{-1}{2}$ **c.** $-\sqrt{3}$ **d.** $\sqrt{2}$ **e.** $\frac{2}{\sqrt{3}}$ **f.** $\frac{1}{\sqrt{3}}$

g. $\frac{-\sqrt{3}}{2}$ **h.** $\frac{-1}{\sqrt{2}}$ **i.** $\sqrt{3}$ **j.** $\frac{-2}{\sqrt{3}}$ **k.** $\sqrt{2}$ **l.** $\sqrt{3}$

m. $\frac{-2}{\sqrt{3}}$ **n.** $\sqrt{2}$ **o.** 1 **p.** 1 **q.** 2

2. $\frac{-(2 + \sqrt{2})}{2}$ **3.** **a.** –1 **b.** 1 **4.** $\frac{2 + \sqrt{2}}{2}$ **5.** $\frac{-(\pi + 2)}{2}$

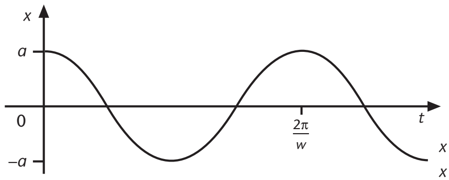
6. **a.** $\frac{7\pi}{6}$ **b.** $\frac{-2}{\sqrt{3}}$ **c.** $\frac{1}{\sqrt{3}}$ **7.** $(-\sqrt{2}, \sqrt{2})$

8. **a.** $\alpha + \theta = \pi$ **(*A*)** **b.** $\theta = 2\pi - \alpha$ **(*A*)** **c.** $\theta = 3\pi - \alpha$ **(*A*)** **d.** $\theta = 3\pi - \alpha$

Investigations: Practical trigonometry (page 14)

1. Answers will vary. **(*M*)** **2.** Tidal transformation $h = H \cos K\pi(t - 3)$ **(*M*)**

3.



$x = a \cos(wt)$
$x = a \sin(\frac{\pi}{2} - wt)$ **(*M*)**

4. **a.** $y = \sin(x + \frac{\pi}{2})$ ie A = 1, B = –1, C = $\frac{\pi}{2}$, D = 0 **(*M*)**

b. $y = -\sin(x - \frac{\pi}{2})$ ie A = 1, B = 1, C = $\frac{-\pi}{2}$, D = 0 **(*M*)**

c. $y = \sin 2(x + \frac{\pi}{4})$ ie A = 1, B = 2, C = $\frac{\pi}{4}$, D = 0 **(*M*)**

d. $y = 2\sin(x + \frac{\pi}{2})$ ie A = 2, B = 1, C = $\frac{-\pi}{2}$, D = 0 **(*M*)**

e. $y = -2\sin 2(x + \frac{\pi}{4})$ ie A = –2, B = 2, C = $\frac{-\pi}{4}$, D = 0 **(*M*)**

f. $y = \frac{1}{2}\sin 3x$ ie A = $\frac{1}{2}$, B = 3, C = 0, D = 0 **(*M*)**

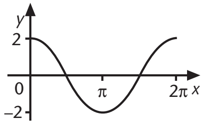
g. $y = 3\sin(-2(x - \frac{\pi}{4})) - 3$ ie A = 3, B = –2, C = $\frac{-\pi}{4}$, D = –3 **(*M*)**

h. $y = 0.5\sin(x - \frac{\pi}{2}) - 2$ ie A = 0.5, B = 1, C = $\frac{-\pi}{2}$, D = –2 **(*M*)**

Note: Other possible values for A, B, C, D may exist.

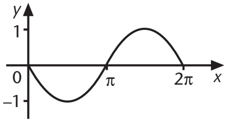
Activity 1.2: Transformations of trigonometric graphs (page 15)

1. a.



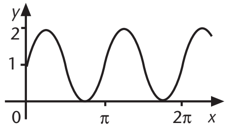
Period = 2π
Amplitude = 2 ***(A)***

b.



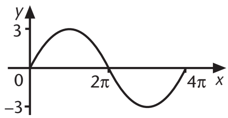
Period = 2π
Amplitude = 1 ***(A)***

c.



Period = π
Amplitude = 1 ***(A)***

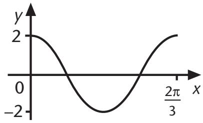
d.



Period = 4π

Amplitude = 3 ***(A)***

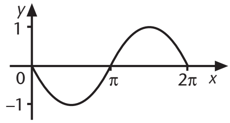
e.



Period = $\frac{2\pi}{3}$

Amplitude = 2 ***(A)***

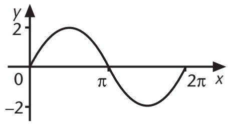
f.



Period = 2π

Amplitude = 1 ***(A)***

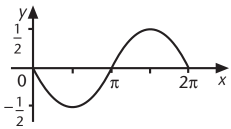
g.



Period = 2π

Amplitude = 2 ***(A)***

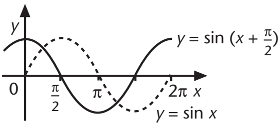
h.



Period = 2π

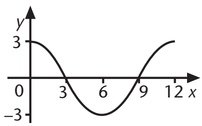
Amplitude = $\frac{1}{2}$ ***(A)***

2. $B = \frac{\pi}{2}$



(A)

3. a.



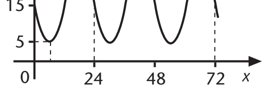
(A)

b. From graph, minimum at (6, –3). ***(A)***

c. 52 minutes and 53 seconds after 8 am;
and 7 minutes and seven seconds after 11 am;
and 52 minutes and 53 seconds past 8 pm;
and 7 minutes and 7 seconds after 11 pm. ***(A)***

4. a. *f*: 3, –3; *g*: 2, –2 ***(A)*** **b.** *f*: 3, *g*: π ***(A)*** **c.** *f*: $\frac{1}{3}$, *g*: $\frac{1}{\pi}$ cycles per hour ***(A)***

5. a.



(A)

b. Maximum temperature = 25°C at 6 pm
Minimum temperature = 5°C at 6 am daily ***(A)***

c. Amplitude = 10; Frequency = $\frac{1}{24}$ cycles per hour ***(A)***

Glossary and Index

absolute value (136, 296): The size of a number or expression, without regard to its sign. Graph **(227)**.

acceleration (197): The rate of change of velocity with respect to time.

amplitude (9): In the graph of a periodic function, the amplitude is the maximum displacement of a point on the graph from the mean or central position.

anticlockwise (2): The opposite direction to the motion of the hands of a clock.

antiderivatives (283): Functions obtained by reversing the process of differentiating.

applications (178, 283, 319, 331): The uses a theory is put to in solving problems.

arbitrary constant (283, 285): A constant of integration which has an undetermined value.

arc (1): Part of a curve. Arc length formula **(36)**.

area of a triangle (36): In any triangle, area = $\frac{1}{2}$ *ab sin* C.

areas under curves (309): The area between a curve and the x-axis is calculated using the definite integral of the equation of the curve. Area between a curve and the y-axis **(311)**. Areas between curves **(313)**.

Argand diagram (110): A complex number plane for graphing complex numbers.

argument (113): The angle, measured anticlockwise, which the line representing a complex number makes with the positive direction of the real axis.

asymptotes (6, 230, 268): Lines which a curve gets closer and closer to but never actually touches.

average velocity (198): The total displacement (distance) divided by the total time taken.

base (66, 154, 157): For indices, the base is the number or letter which is raised to a power, eg in the expression a^x, the base is a. In a logarithm expression such as $\log_b x$, the base is b.

base changing rule (69): A rule for changing logarithms from one base to another.

binomial (90): An expression with two terms.

Binomial Theorem (94): A rule for expanding $(a + b)^n$.

bracket rule (147): A specific case of the chain rule, used for differentiating brackets raised to powers.

Cartesian equation (170): An equation connecting two variables x and y from which a graph can be drawn.

Cartesian form (113): See *rectangular form*.

chain rule (146, 203): A rule for differentiating composite functions.

chord (251): A line segment whose end points are on a curve.

circle (227, 233, 235–240): Curve with equation of the form $x^2 + y^2 = R^2$.